Integral high gain observers: Theory and experimental evaluation

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Abstract. Integral High Gain Observer evaluation is performed for the simultaneous estimation of the state variables and of an uncertain term for a class of nonlinear systems. It is assumed that the measured output presents additive noise. The proposed observer contains an integral-type contribution of the measured output, which provides robustness against noise. The observer is applied to a 1 degree of freedom (DOF) robot. Experimental results in real time confirm the validity and good performance of the observer.

1 Introduction

Thanks to the advances in the automatic control field, it has been possible to develop methods and techniques to estimate the unknown variables governing the behaviour of a dynamical system. A technique commonly employed for this purpose is based on the reconstruction of the required variables by means of an *observer*, which is an auxiliary dynamical system that attempts to *estimate* and *reconstruct* the state variables by using the model of the original system, its input and output. The first developments in estimation theory were obtained by Wiener [8] about temporal series analysis, based on statistical assumptions related to the nature of the noise present in the signal to be filtered and developing the so-called *Wiener filter*. A major thrust in the development of observers came from the introduction of state-space methods by R.E. *Kalman* in 1960 [7]. In 1963 D.G. *Luenberger* [7] showed that for observable linear systems, an observer can be designed with the property that the observation error, generated by the difference between the original system measured output and the output estimated by the observer, goes exponentially to zero [2], [7]. In order to implement a Luenberger observer it is necessary to know exactly the structure and the parameters of the system under observation. In real-life systems, it is unlikely to exactly known the above information then precluding the use of Luenberger observers at least in its theoretical form. Several modifications to the original Luenberger observer have been proposed to cope with the above problem [1], [2], [3], [4].

The integral observers proposed in [1] are particularly interesting for several reasons. They are high gain designs in which the observer gain is tuned using a single parameter. Moreover, their structure allows estimating certain classes of uncertainties in the same way as it is showing in [6].

Another interesting feature of integral observers is the fact that they are driven by the integral of the system output whereas in standard observers the observer is drivendirectly by the system output. As a consequence, measurement noise is filtered by the integral action then allowing using high gain in practice. Unfortunately, the integral-type observers have been evaluated using only numerical simulations and no study of its implementation in real-time has not been conducted.

The aim of this work is to experimentally evaluate a class of Integral-type high gain observers employing an electromechanical laboratory prototype consisting of a 1 DOF robot driven by a Direct Current (DC) motor. Two cases are considered. Firstly, it is assumed that all the prototype parameters are known, secondly, an uncertainty on the robot model is considered. The paper is presented as follows. Section 2 is devoted to the integral-type high-gain observers. Application of the observer presented in Section 2 to a 1 DOF robot is shown in Section 3. Section 5 presents the experimental results for the two cases discussed above, namely, design of the observer with and without uncertainties. The paper ends with some concluding remarks.

2 Integral High Gain Observers

The system studied in next section belongs to a class of partially known nonlinear systems described by

$$\begin{aligned} \dot{X} &= \mathrm{L}(X,U) + \mathrm{J}(X) \\ Y &= CX + d \end{aligned} \tag{1}$$

where $X \in \mathbb{R}^n$ is the state vector, $U \in \mathbb{R}^m$ is the control input vector, $\mathrm{J}(\cdot)$ is a nonlinear partially known vector, $\mathrm{L}(\cdot)$ is a linear vector of its arguments, $d \in \mathbb{R}$ is a bounded measurement additive noise and $Y \in \mathbb{R}$ is the system output.

Now, consider the following assumptions

A1. $d \in \mathbb{R}$ represents an external bounded additive noise

$$|d| \le \alpha, \qquad 0 < \alpha < \infty .$$

A2. The system given by (1) is locally uniformly observable [5].

The following representation of (1) is proposed [1]

$$\begin{aligned} \dot{X}_0 &= CX + d \\ \dot{X} &= \mathrm{L}(X,U) + \mathrm{J}(X) \\ \dot{\mathrm{J}} &= \Theta(X) \\ Y_0 &= X_0 \end{aligned} \tag{2}$$

A key observation about (2) is the fact that the uncertain term J is considered as a new state and $\Theta(X)$ is a nonlinear unknown function that describes the dynamics of J, and satisfies the following assumption

A3. The dynamics $\Theta(X)$ of the uncertain term J is bounded, therefore

$$\|\Theta(X)\| \leq \beta, \qquad \beta > 0 .$$

Another important remark is that another state x_0 is added by integrating the system output.
Then, the following dynamic system is an integral observer for (2) [1]

$$\begin{aligned} \dot{\hat{X}}_0 &= C\hat{X} + k_1\left(Y_0 - \hat{Y}_0\right) \\ \dot{\hat{X}} &= \mathrm{L}\left(\hat{X}, U\right) + \mathrm{J}\left(\hat{X}\right) + k_2\left(Y_0 - \hat{Y}_0\right) \\ \dot{\hat{\mathrm{J}}} &= k_3\left(Y_0 - \hat{Y}_0\right) \\ \hat{Y}_0 &= \hat{X}_0 \end{aligned} \tag{3}$$

The observer gain matrix $K = [k_1, k_2, k_3]^T$ is given as

$$K = S_\theta^{-1} C^T . \tag{4}$$

S_θ is a symmetric positive definite solution of the algebraic Lyapunov equation [5]

$$S_\theta\left(A + \frac{\theta}{2} I\right) + \left(A^T + \frac{\theta}{2} I\right) S_\theta = C^T C . \tag{5}$$

Where

$$C = \begin{bmatrix}1 & 0 \cdots\cdots 0\end{bmatrix} \text{ and } A_{ij} = \begin{cases} 1, & \text{if } j = i+1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

For some parameter $\theta > 0$, determining the desired convergence speed. Coefficients of S_θ are given by $(S_\theta)_{ij} = \frac{s_{ij}}{\theta^{i+j-1}}$, where s_{ij} are entries of a symmetric positive definite matrix not depending on θ.

3 System Under Study

Let the single input-single output nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k\sin(x_1) - ax_2 + bu \\ y = x_1 + d \end{cases} \tag{7}$$

corresponding to a 1 DOF robot model affected by additive measurement noise d.

3.1 Totally Known System

According to the results presented in Section 2, a new representation of the system (7) is proposed. Set $x_0(t)=\int_0^t y(\tau)d\tau$ so that $\dot{x}_0(t)=y(t)=x_1(t)+d$. Then, the following augmented system is obtained

$$\begin{cases} \dot{x}_0 = x_1 + d \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = -k\sin(x_1) - ax_2 + bu \\ y_0 = x_0 \end{cases} \tag{8}$$

(8) can be written in matrix form as follows

$$\begin{aligned} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -(k\sin(x_1)+ax_2) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d \\ y_0 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{9}$$

Or in a compact form

$$\begin{cases} \dot{x} = Ax + Bu + \Phi(x) + Ed \\ y_0 = Cx \end{cases} \tag{10}$$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix};\ \Phi(x) = \begin{bmatrix} 0 \\ 0 \\ -(k\sin(x_1)+ax_2) \end{bmatrix};\ E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Notice that $E=C^T$ and the matrix A and C of the new augmented system (8) retain the properties required in (6), as well as system (10) is observable.
An integral high-gain observer design for (10) results in

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + \Phi(\hat{x}) + K(y_0 - \hat{y}_0) \\ \hat{y}_0 = C\hat{x} \end{cases} \tag{11}$$

Combining (10) and (11) the estimation error dynamics $e = x - \hat{x}$ is given as

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = (A - KC)e + \Phi(x) - \Phi(\hat{x}) + Ed \\ \dot{e} &= A_\theta e + \Phi + C^T d \end{aligned} \tag{12}$$

With K defined as in (4), $A_\theta = A - KC$ and $\Phi = \Phi(x) - \Phi(\hat{x})$.

From (12) can be seen that the noise is not amplified by the observer gain. S_θ , S_θ^{-1} and A_θ matrices for (11) and (12) are given by

$$S_\theta = \begin{bmatrix} \frac{1}{\theta} & -\frac{1}{\theta^2} & \frac{1}{\theta^3} \\ -\frac{1}{\theta^2} & \frac{2}{\theta^3} & -\frac{3}{\theta^4} \\ \frac{1}{\theta^3} & -\frac{3}{\theta^4} & \frac{6}{\theta^5} \end{bmatrix}; \quad S_\theta^{-1} = \begin{bmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{bmatrix}; \quad A_\theta = \begin{bmatrix} -3\theta & 1 & 0 \\ -3\theta^2 & 0 & 1 \\ -\theta^3 & 0 & 0 \end{bmatrix}.$$

Observer (11) for system (10) has the following structure

$$\begin{cases} \dot{\hat{x}}_0 = \hat{x}_1 + 3\theta(x_0 - \hat{x}_0) \\ \dot{\hat{x}}_1 = \hat{x}_2 + 3\theta^2(x_0 - \hat{x}_0) \\ \dot{\hat{x}}_2 = -k\sin(\hat{x}_1) - a\hat{x}_2 + bu + \theta^3(x_0 - \hat{x}_0) \end{cases} \tag{13}$$

3.2 Uncertainty Estimation

Based on (7), several terms were grouped in a single term and considered as uncertain, i.e. $-(k\sin(x_1) + ax_2) = \varphi(x,u)$ is the uncertainty to be estimated. Then, system (8) can be rewritten as

$$\begin{cases} \dot{x}_0 = x_1 + d \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = \varphi(x,u) + bu \\ y_0 = x_0 \end{cases} \tag{14}$$

Incorporating the uncertainty to the system as a new state the following augmented system is obtained

$$\begin{cases} \dot{x}_0 = x_1 + d \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 + bu \\ \dot{x}_3 = \Theta(x,u) \\ y_0 = x_0 \end{cases} \tag{15}$$

Where $\Theta(x,u)$ is an unknown nonlinear function describing the dynamics of $x_3 = \varphi(x,u)$. The matrix form of (15) is given by

$$\begin{bmatrix}\dot{x}_0\\ \dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_0\\x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\b\\0\end{bmatrix}u+\begin{bmatrix}0\\0\\0\\\Theta(x)\end{bmatrix}+\begin{bmatrix}1\\0\\0\\0\end{bmatrix}d \tag{16}$$
$$y_0=\begin{bmatrix}1&0&0&0\end{bmatrix}\begin{bmatrix}x_0&x_1&x_2&x_3\end{bmatrix}^T$$

Which in a compact form is represented as

$$\begin{cases}\dot{x}=Ax+Bu+\Psi(x)+Ed\\ y_0=Cx\end{cases} \tag{17}$$

Where

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix};\quad B=\begin{bmatrix}0\\0\\b\\0\end{bmatrix};\quad \Psi(x)=\begin{bmatrix}0\\0\\0\\\Theta(x)\end{bmatrix};\quad E=\begin{bmatrix}1\\0\\0\\0\end{bmatrix};\quad C=\begin{bmatrix}1&0&0&0\end{bmatrix}.$$

Again $E=C^T$ and the matrix A and C of the new augmented system (15) conserve the properties required in (6). An integral high-gain observer for (17) is given by

$$\begin{cases}\dot{\hat{x}}=A\hat{x}+Bu+K(y_0-\hat{y}_0)\\ \hat{y}_0=C\hat{x}\end{cases} \tag{18}$$

Using (17) and (18), the estimation error dynamics $e=x-\hat{x}$ is given as follows

$$\begin{aligned}\dot{e}=\dot{x}-\dot{\hat{x}}=(A-KC)e+\Psi(x)+Ed\\ \dot{e}=A_\theta e+\Psi(x)+C^T d\end{aligned} \tag{19}$$

With K defined as in (4). Matrices S_θ, S_θ^{-1} and A_θ for (18) and (19) are given by

$$S_\theta=\begin{bmatrix}\frac{1}{\theta}&-\frac{1}{\theta^2}&\frac{1}{\theta^3}&-\frac{1}{\theta^4}\\ -\frac{1}{\theta^2}&\frac{2}{\theta^3}&-\frac{3}{\theta^4}&\frac{4}{\theta^5}\\ \frac{1}{\theta^3}&-\frac{3}{\theta^4}&\frac{6}{\theta^5}&-\frac{10}{\theta^6}\\ -\frac{1}{\theta^4}&\frac{4}{\theta^5}&-\frac{10}{\theta^6}&\frac{20}{\theta^7}\end{bmatrix}\quad S_\theta^{-1}=\begin{bmatrix}4\theta&6\theta^2&4\theta^3&\theta^4\\6\theta^2&14\theta^3&11\theta^4&3\theta^5\\4\theta^3&11\theta^4&10\theta^5&3\theta^6\\\theta^4&3\theta^5&3\theta^6&\theta^7\end{bmatrix}\quad A_\theta=\begin{bmatrix}-4\theta&1&0&0\\-6\theta^2&0&1&0\\-4\theta^3&0&0&1\\-\theta^4&0&0&0\end{bmatrix}$$

The observer (18) for the system (17) has the following structure

$$\begin{cases} \dot{\hat{x}}_0 = \hat{x}_1 + 4\theta\left(x_0 - \hat{x}_0\right) \\ \dot{\hat{x}}_1 = \hat{x}_2 + 6\theta^2\left(x_0 - \hat{x}_0\right) \\ \dot{\hat{x}}_2 = \hat{x}_3 + bu + 4\theta^3\left(x_0 - \hat{x}_0\right) \\ \dot{\hat{x}}_3 = \theta^4\left(x_0 - \hat{x}_0\right) \end{cases} \tag{20}$$

4 Stability Analysis

Let the following error dynamics

$$\dot{e} = A_\theta e + \Omega(X,U,d)\ . \tag{21}$$

Where

$$e = \begin{bmatrix} X_0 - \hat{X}_0 \\ X - \hat{X} \\ \mathrm{J} - \hat{\mathrm{J}} \end{bmatrix};\quad \Omega = \begin{bmatrix} d \\ \Delta\mathrm{L} \\ \Theta(X) \end{bmatrix};\quad A_\theta = \left(A - S_\theta^{-1} C^T C\right).$$

And consider the following assumptions

A4. The function $\Delta\mathrm{L} = \mathrm{L}(X,U) - \mathrm{L}\left(\hat{X},U\right)$ is bounded, i.e.

$$\left\|\mathrm{L}(X,U) - \mathrm{L}\left(\hat{X},U\right)\right\| \le \delta, \quad \delta > 0\ .$$

A5. There exist two positive constants $\gamma > 0$ and $\lambda > 0$ satisfying

$$\left\|\exp(A_\theta t)e\right\| \le \gamma \exp(-\lambda t)\left\|e\right\|\ .$$

A6. The vector function Ω is bounded, so that

$$\left\|\Omega\right\| \le \mu, \qquad \mu > 0\ .$$

Now, by solving (21), the following expression is obtained

$$e = \exp(A_\theta t)e(0) + \int_0^t \exp\left\{A_\theta\left(t-\tau\right)\right\}\Omega d\tau\ . \tag{22}$$

Considering assumptions *A1* to *A6* and taking norms for both sides of (22), the following inequality is obtained

$$\left\|e\right\| \le \gamma\exp(-\lambda t)\left[\left\|e(0)\right\| - \frac{\mu}{\lambda}\right] + \frac{\gamma\mu}{\lambda}\ . \tag{23}$$

Taking the limit when $t \to \infty$, the following bound on e is obtained

$$\|e\| \leq \frac{\gamma\mu}{\lambda} . \tag{24}$$

Then, the estimation error remains bounded inside a ball of radius $\gamma\mu / \lambda$, i.e. $e \in B_{\frac{\gamma\mu}{\lambda}}(0)$.

5 Experimental Results

In order to perform the experiments, a 1 DOF robot was employed and it is shown in Fig. 1.

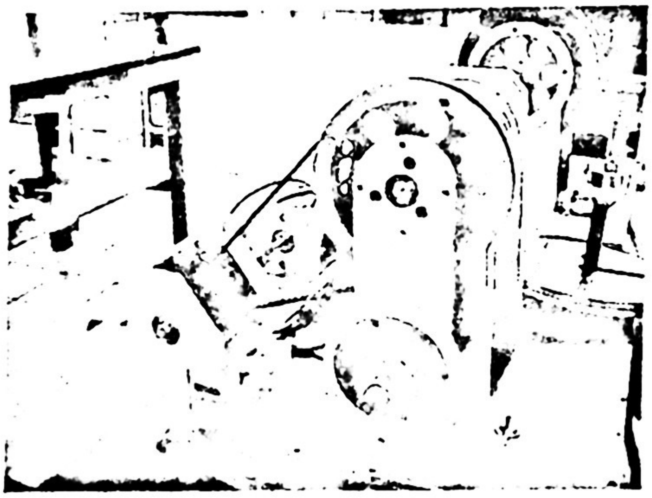

Fig. 1. Laboratory prototype employed in the experiments

A DC-motor drives the link through a belt and angular position is measured using an optical encoder with 2500 pulses per turn. The encoder is directly attached to the link. The motor is driven by a Copley Controls, model 413, power amplifier configured in torque mode. Data acquisition is performed using the MultiQ 3 card from Quanser Consulting with optical encoder inputs. These inputs multiply by 4 the encoder resolution, then, the number of pulses per turn is 10000. This output was further scaled down by a factor of 10000 corresponding to one link turn. The card also has 12 bits Digital to Analog converters with an output voltage range of $\pm 5v$. All the programming including the observers and data acquisition was implemented using the MatLab-Simulink software running under the WinCon program from Quanser Consulting. The WinCon environment was used in the client and the server running on different computers. The server is installed in a Pentium based computer running at $200 Mhz$. The client is allocated in another Pentium based computer running at $350 Mhz$. Sampling rate was set to $5 Khz$.

Since the 1 DOF robot may get unstable in open loop, a loop was closed around the robot using a relay programmed in the MatLab-Simulink environment. In this way, the robot performed a stable limit cycle. The velocity estimated from the observers was compared with another velocity estimate obtained from position measurements through numerical differentiation. Parameters for model (7) are $a = 0.7$, $b = 28$ and $k = 12.04$.

First Case: Robot without Uncertainty.
In this case the nonlinear terms in (7) are assumed known. Observer gain was set to $\theta = 120$. Fig. 2 and Fig. 3 show position and velocity from the real system and from the observer. It is noted that measured and estimated position do not show noticeable differences. On the other hand, velocity estimates show some variation. The estimate obtained from the integral observer shows a smoother behaviour compared with the velocity estimated obtained from numerical differentiation of the position measurement, a technique of broad use in industrial servomotors.

Second Case: Robot with Uncertainty.
In this case observer gain was set to $\theta = 80$. Fig. 4 and Fig. 5 display the position and velocity estimates. Behaviour in both states is similar to the behaviour observed in Fig. 2 and Fig. 3. Fig. 6 show the real and estimated uncertainty, the later exhibiting a smoother behaviour. The uncertainty estimate does not match perfectly the real uncertainty, however, it should be pointed out that the so called real uncertainty is also an estimate since it was computed using estimated parameters, position measurements possibly contaminated with noise and velocity estimates obtained from position measurements through numerical differentiation. Moreover, it is worth remarking that the integral observer does not need any a priory information regarding the uncertainty structure and parameters. Note also that even with very high observer gains, measurement noise does not affect the estimates. In the particular case of uncertainty estimation, the corresponding gain was $\theta^4 = 40960000$.

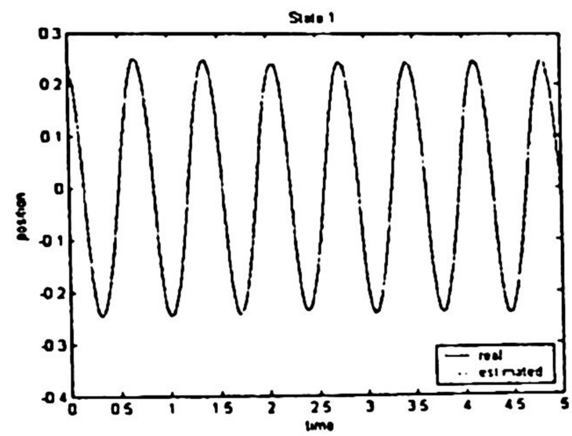

Fig. 2. Position

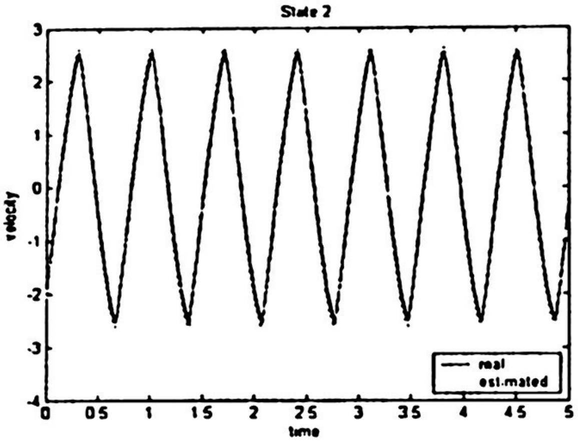

Fig. 3. Velocity

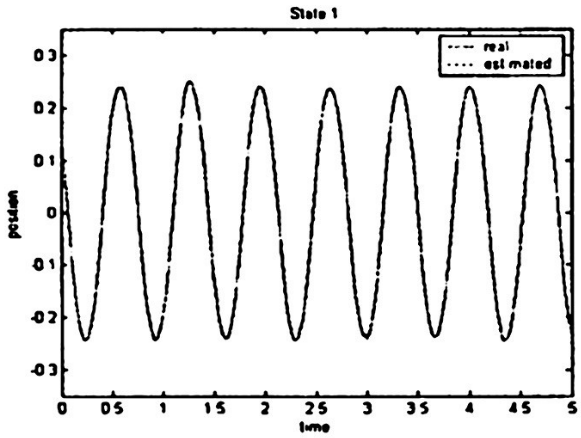

Fig. 4. Position

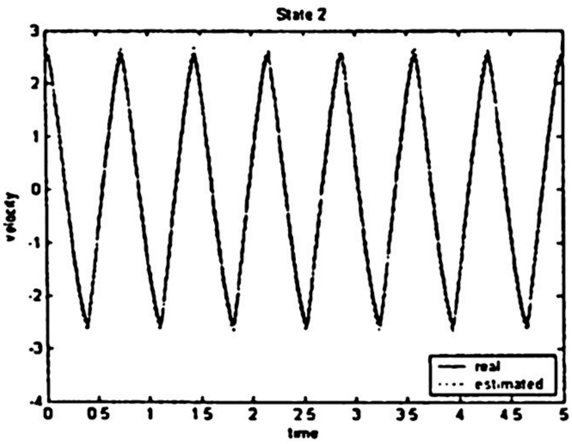

Fig. 5. Velocity

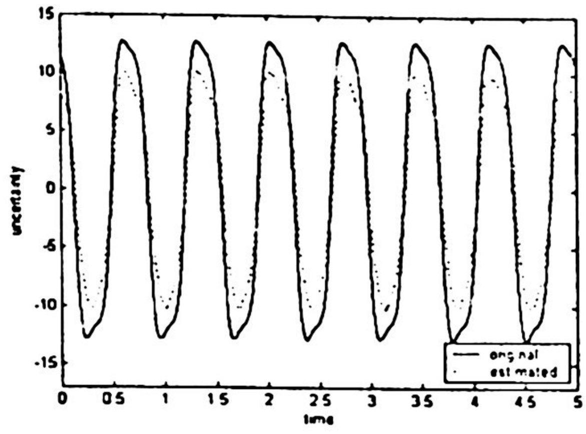

Fig. 6. Uncertainty

6 Conclusions

Few works regarding experimental evaluation of high gain observers have been published in the open literature. In this work and for the first time, an Integral-type high gain observer was experimentally evaluated. A feature of this observer is the fact that it is driven by the integral of the system output, an approach contrasting with the standard method in which the system output drives directly the observer. As a consequence, measurement noise is highly attenuated then allowing using very high observer gains. Performance of the integral observer was evaluated through experiment using a 1 DOF robot. Two cases were studied. In the first one all the robot parameters are assumed known. In the second case, the robot nonlinear term and viscous friction were supposed unknown.

In both cases the observer showed good performance in spite of the high gain employed.

Future work includes using the Integral observer in closed loop with a Proportional-Derivative control law applied to the same prototype.

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